Concordance Lecture 3

September 9, 2021 1:35 PM

- [ ] Mic check
- [ ] Record
- [ ] Next lecture online
- [ ] Problem sessions

Last week: slice, ribbon, concordance

This week: invariants and algebraic concordance

1 SEIFERT FORM

Recall A **Seifert surface** F for a knot K is a compact, connected, oriented surface with $\partial F = K$.

- always exist by Seifert's algorithm
- can usually just draw them
- matches orientation of K

Note F always has a product nbhd $F \times [-1,1] \subseteq S^3$

Linking # $lk(K_1, K_2) =$ sum of signed (transverse) intersections between K_1 and a Seifert surface for K_2

Example

$lk(K_1, K_2) = 1$

$lk(\bigcirc\bigcirc) = 0$

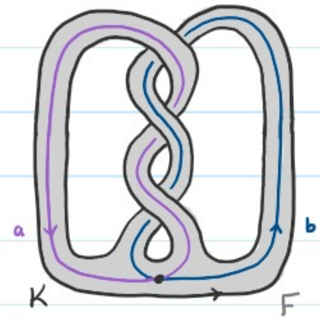
$lk($  $) = 0$

See Rolfsen 5D, p132

- for more info and equivalent defn's (lk_2 ours, lk_3 nice)
- is a concordance invariant of links
- 4D version: sum of signed (transverse) intersection between oriented surfaces F_1 and F_2 in B^4 with $\partial F_i = K_i$ (note if $F_1 \cap F_2 = \emptyset$, $lk = 0$)

Seifert Pairing Fix knot K and Seifert surf. F

Recall $H_1(F;\mathbb{Z}) = \mathbb{Z}^{2g}$, g = genus of F



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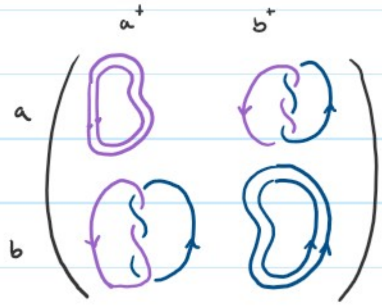


$\xrightarrow{lk}$ $V = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

For curve γ on F, let $\gamma^{\pm} = \gamma \times \{\pm 1\} \subseteq F \times [-1,1]$

Defn The **Seifert pairing** of F is the map

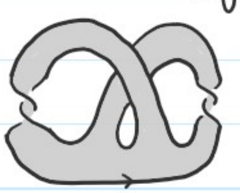
$$V : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

$$([x],[y]) \longmapsto lk(x, y^+)$$

With a chosen basis, the corresponding $2g \times 2g$ matrix is the **Seifert matrix** V.

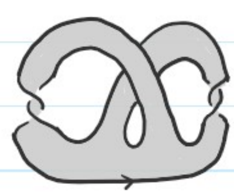
Examples

trefoil



$V_3 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

figure-eight



$V_4 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

WARNING Not a knot invariant, even under congruence

$V \sim W \iff V = PWP^T$ some invtble P

UPSHOT Easier to work with matrices than directly with $\mathcal{C}$

(Want homomorphisms $\mathcal{C} \longrightarrow G$ for some nice G like $\mathbb{Z}$)

Defn For a knot K with Seifert matrix S,

① **Alexander polynomial**: $\Delta_t(K) = \det(V - tV^T)$ a Laurant poly considered up to mult by $\pm t^n, n \in \mathbb{Z}$ (see Rolfsen 8C)

$$\Delta_t(3_1) = \det\begin{pmatrix} -1+t & 1 \\ -t & -1+t \end{pmatrix} = t^2 - t + 1 \qquad \Delta_t(4_1) = \begin{pmatrix} -1+t & 1 \\ -t & 1-t \end{pmatrix} = -t^2 + 3t - 1$$

② **Signature**: $\sigma(K)$ = signature of $V + V^T$ over $\mathbb{R}$ (see Rolfsen 8E1 – 8E3)

$$V_3 \sim \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Longrightarrow \sigma(3_1) = 2 \qquad V_4 \sim \begin{pmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{pmatrix} \Rightarrow \sigma(4_1) = 0$$

FACT Both are independent of choices (V, basis, etc)
ie. are knot invariants (see S-equivalence)

Our goal is to prove:

Thm If K is topologically slice, then
(a) Fox-Milnor condition: $\Delta_t(K) \doteq f(t)f(t^{-1})$ (over $\mathbb{Z}$)
(b) $\sigma(K) = 0$

Proof sketch: K top slice $\Rightarrow V = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix} \xrightarrow{\text{HW}}$ (a), (b)

* Gives us sliceness obstructions
* Hard(ish) to tell if a poly factors

Defn The **determinant** of a knot K is $\det(K) := \det(V + V^T)$.

Note $\det(K) = |\Delta_{-1}(K)|$

Note K top slice $\Rightarrow \det(K)$ is a square

Ex's
$\sigma(3_1) = 2$, $\det(3_1) = 3$
$\sigma(4_1) = 0$, $\det(4_1) = 5$
$\Rightarrow$ both not slice

Cor σ is a concordance inv't

$K \sim J \iff K \# -J$ is slice

$\implies 0 = \sigma(K \# -J) \overset{HW}{=} \sigma(K) + \sigma(-J) \overset{HW}{=} \sigma(K) - \sigma(J)$

$\implies \sigma(K) = \sigma(J)$

! By above ex's $[3_1] \in \mathcal{C}^{TOP}$ has ∞-order **FACT** $[4_1]$ has order 2

2 Algebraic Concordance Group

Goal: Construct surj $\mathcal{C} \twoheadrightarrow \mathcal{G}$ due to J. Levine (1969)

idea for $\mathcal{G}$ is that Seif. matrices were so nice we should just work with these

Defn A $2g \times 2g$ matrix is *hyperbolic* if it is congruent to a block matrix $\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$

Defn The *block sum* of square matrices A, B is $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

(K is concordant to J iff $K \# -J$ is slice)

Defn A is *cobordant* to B if $A \oplus -B$ is hyperbolic

exercise: congruent $\Rightarrow$ cobordant

(HW) exercise: reflexive and symmetric

if $\det(A - A^T) \neq 0$, then also transitive

Defn The *algebraic concordance group* is the set

$$\mathcal{G} = \left\{ \begin{matrix} \text{square integral matrices} \\ \text{w/ } \det(A - A^T) = \pm 1 \end{matrix} \right\} \Big/ \text{cobordism} \quad \text{under } \oplus$$

Thm $(\mathcal{G}, \oplus)$ is an abelian group

Proof in HW: identity [hyperbolic mat's]

inverses $-[A] := [-A]$

Theorem (Levine '69) $\mathcal{C} \xrightarrow{\psi} \mathcal{G}$ is a surj.

$[K] \longmapsto [V]$

Two things to check:

- well-defined
- surjective (constructive, see Livingston-Naik 1.8.13)

Let K, J have Seif mats V, W

WTS $K \sim J \Rightarrow V$ cobordant to W

V cob W iff $V \oplus -W$ is congruent to a hyperbolic mat

Seif form for $K \# -J$ slice!

So it suffices to show:

$\star$ Slice knots admit hyperbolic Seifert forms